Brackets in the jet-bundle approach to field theory

G. Barnich

Freie Universität Berlin, Fachbereich Physik, Institut für Theoretische Physik, Arnimallee 14, D-14195 Berlin.

Abstract

In the first part the sh Lie structure of brackets in field theory, described in the jet bundle context along the lines suggested by Gel'fand, Dickey and Dorfman, is analyzed. In the second part, we discuss how this description allows us to find a natural relation between the Batalin-Vilkovisky antibracket and the Poisson bracket.

(*)Alexander-von-Humboldt fellow. On leave of absence from Chargé de Recherches du Fonds National Belge de la Recherche Scientifique at Univerité Libre de Bruxelles.

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1 Sh Lie structure of brackets on the horizontal complex

1.1 The horizontal complex as a resolution for local functionals

In the approach of Gel'fand, Dickey and Dorfman to functionals in field theory [GeDi1, GeDi2, GeDo1, GeDo2, GeDo3] (see [Dic1] for a review), one replaces local functionals satisfying appropriate boundary conditions by equivalence classes of local functions.

Let M be an n-dimensional manifold homeomorphic to bfR^n with coordinates denoted by x^i and $\pi : E = M \times V \to M$ a trivial vector bundle of fiber dimension k over M. The coordinates of V are denoted by u^a.

Let $J^\infty E$ denote the infinite jet bundle of E over M with $\pi^\infty_E : J^\infty E \to E$ and $\pi^\infty_M : J^\infty E \to M$ the canonical projections. The vector space of smooth sections of E with compact support will be denoted ΓE. For each section ϕ of E, let $j^\infty\phi$ denote the induced section of the infinite jet bundle $J^\infty E$. The bundle

$$\pi^\infty : J^\infty E = M \times V^\infty \to M \tag{1.1}$$

then has induced coordinates given by

$$(x^i, u^a, u^a_i, u^a_{i_1 i_2}, \dots,). \tag{1.2}$$

Definition 1.1 *A local function on $J^\infty E$ is the pullback of a smooth function on some finite jet bundle $J^p E$, i.e., a composite $J^\infty E \to J^p E \to bfR$. In local coordinates, a local function $L(x, u^{(p)})$ is a smooth function in the coordinates x^i and the coordinates u^a_I, where the order $|I| = r$ of the multi-index I is less than or equal to some integer p. The space of local functions will be denoted by $Loc(E)$*

Let ν denote a fixed volume form on M and let ν also denote its pullback $(\pi^\infty_E)^*(\nu)$ to $J^\infty E$. In coordinates, $\nu = d^n x = dx^1 \wedge \cdots \wedge dx^n$.

Definition 1.2 *A local functional*

$$\mathcal{L}[\phi] = \int_M L(x, \phi^{(p)}(x))\nu = \int_M (j^\infty \phi)^* L(x, u^{(p)})\nu \tag{1.3}$$

is the integral over M of a local function evaluated for sections ϕ of E of compact support. The space of local functionals $\mathcal{F}$ is the vector space of equivalence classes of local functionals, where two local functionals are equivalent if they agree for all sections of compact support.

The total derivative D_i along x^i is

$$D_i = \frac{\partial}{\partial x^i} + u^a_{iJ} \frac{\partial}{\partial u^a_J}. \tag{1.4}$$

The horizontal complex (Ω^*, d_H) is the exterior algebra in the dx^i with coefficients that are local functions and differential $d_H = dx^i D_i$.

Lemma 1.1 *The vector space of local functionals $\mathcal{F}$ is isomorphic to the cohomology group $H^n(d_H)$.*

Proof. A complete proof can be found for instance in [Olv]. $\square$.

Furthermore, one can also show (see for instance [Olv]) that the horizontal complex without the constants provides a homological resolution of $\mathcal{F}$, i.e., the complex

$$\begin{array}{ccccccccc}
\Omega^0/bfR & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-2} & \xrightarrow{d_H} & \Omega^{n-1} & \xrightarrow{d_H} & \Omega^n \\
\downarrow \eta & & & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\
0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_0 = \mathcal{F}
\end{array}$$

is exact.

1.2 Algebraic definition of brackets

In the simplest case, brackets in field theory are defined by splitting the coordinates u^a of an even-dimensional V in two sets u^α and π_β and declaring these two be canonically conjugate, i.e., $\{u^\alpha(x), \pi_\beta(y)\} = \delta^\alpha_\beta \delta(x,y)$, while all other brackets vanish. Brackets of local functionals are then defined in terms of functional derivatives by

$$\{\mathcal{L}_1, \mathcal{L}_2\} = \int_M \frac{\delta \mathcal{L}_1}{\delta u^\alpha(x)} \frac{\delta \mathcal{L}_2}{\delta \pi_\beta(x)} \nu - (1 \longleftrightarrow 2). \tag{1.5}$$

In the case of sections with compact support, the functional derivative is equal to the Euler-Lagrange derivative pulled back to the corresponding section:

$$\frac{\delta \mathcal{L}_1}{\delta u^\alpha(x)} = (j^\infty \phi)^* \frac{\delta L_1}{\delta u^\alpha}, \tag{1.6}$$

the Euler-Lagrange operator being defined by

$$\frac{\delta L_1}{\delta u^a} = \frac{\partial L_1}{\partial u^a} - \partial_i \frac{\partial L_1}{\partial u^a_i} + \partial_i \partial_j \frac{\partial L_1}{\partial u^a_{ij}} - ... = (-D)_I (\frac{\partial L_1}{\partial u^a_I}). \tag{1.7}$$

This suggest taking

$$\{L_1 \nu, L_2 \nu\} = [\frac{\delta L_1}{\delta u^\alpha} \frac{\delta L_2}{\delta \pi_\beta} - (1 \longleftrightarrow 2)] \nu \tag{1.8}$$

as a definiton of a bracket on Ω^n. Since Euler-Lagrange derivatives annihilate total divergences, we have

$$\{d_H R, L_2 \nu\} = 0 \tag{1.9}$$

for $R \in \Omega^{n-1}$, so that there is indeed a well defined induced bracket on $\mathcal{F} \simeq H^n(d_H)$, given by

$$\begin{aligned} \{\cdot, \cdot\} : \mathcal{F} \wedge \mathcal{F} \longrightarrow \mathcal{F}, \\ \{[L_1 \nu], [L_2 \nu]\} = [\{L_1 \nu, L_2 \nu\}]. \end{aligned} \tag{1.10}$$

As in the case of standard mechanics, brackets need not be defined in terms of Darboux coordinates as above. Let $TDO(E)$ be the space of total

differential operators. In coordinates, a total differential operator is given by an operator of the form $Z^I D_I$, with $Z^I \in Loc(E)$. Let ω^{ab} be a square matrix of total differential operators, $\omega^{ab} = \omega^{abI} D_I$. We then have the following general definition (see e.g. [Olv, Dic1]) :

Definition 1.3 *The operator matrix $\omega^{ab} \in TDO(E)$ defines a skew-symmetric bracket on Ω^n through*

$$\{\cdot,\cdot\} : \Omega^n \wedge \Omega^n \longrightarrow \Omega^n,$$

$$\{L_1\nu, L_2\nu\} = \frac{1}{2}[\omega^{ab}(\frac{\delta L_1}{\delta u^a})\frac{\delta L_2}{\delta u^b} - (1 \longleftrightarrow 2)]\nu. \tag{1.11}$$

In principle, one could allow for an arbitrary combination of the Euler-Lagrange derivatives, $\omega^{abIJ} D_I \frac{\delta L_1}{\delta u^a} D_J \frac{\delta L_2}{\delta u^b}$, with $\omega^{abIJ} \in Loc(E)$. Since we are only interested in the brackets induced in $\mathcal{F}$, this case can be reduced to the above by integrations by parts giving d_H exact terms projecting to zero in $\mathcal{F}$. More precisely, one can show [Olv] that a skew-symmetric bracket on $\mathcal{F}$ is uniquely determined by a functional two vector which in turn is uniquely determined by a skew-adjoint matrix of total differential operators $\omega^{ab} : \omega^{abI} D_I f_a = -(-D)_I[\omega^{baI} f]$ for all $f \in Loc(E)$.

1.3 Jacobi identity

The requirement that the induced bracket on $\mathcal{F}$ satisfies the Jacobi identity is equivalent to

$$\{\{L_1\nu, L_2\nu\}, L_3\nu\} + cyclic = d_H R(L_1\nu, L_2\nu, L_3\nu) \tag{1.12}$$

for all $L_1\nu, L_2\nu, L_3\nu \in \Omega^n$ with $R(L_1\nu, L_2\nu, L_3\nu) \in \Omega^{n-1}$. This imposes restrictions on the skew-adjoint total differential operator ω^{ab}. If these restrictions are satisfied, one says [Olv, Dic1] that ω^{ab} is Hamiltonian. The simplest examples of Hamiltonian ω^{ab}'s are given by the constant symplectic matrix σ^{ab} considered above in the case of the bracket in Darboux coordinates, as well as the total differential operator D_x of the KDV equation, where the dimension n of the base space and k of the fiber are both equal to 1 (see also [Dic2]).

We thus see that in the case where the induced bracket is a Lie bracket, the bracket $l_2 \equiv \{\cdot,\cdot\}$ on Ω^n induces a completely skew-symmetric bracket

with three entries defined by

$$\begin{aligned} l_3 : \wedge^3\Omega^n &\longrightarrow \Omega^{n-1} \\ l_3(L_1\nu, L_2\nu, L_3\nu) &= R(L_1\nu, L_2\nu, L_3\nu). \end{aligned} \tag{1.13}$$

1.4 Sh Lie algebras from homological resolutions of Lie algebras

The algebraic situation we have is the following. Let $\mathcal{F}$ be a vector space and (X_*, l_1) be a homological resolution thereof, i.e., we have a graded vector space $X_* = X_0 \oplus X_1 \oplus \dots$, with a differential $l_1 : X_k \longrightarrow X_{k-1}$, such that $H_0(l_1) \simeq \mathcal{F}$ and $H_k(l_1) = 0, k > 0$.

(In our case above, $\mathcal{F}$ is the space of local functionals, X_* is the horizontal complex without the constants, if we define the grading to be n minus the horizontal form degree and identify $l_1 = d_H$.)

Let $[\cdot,\cdot]$ be a homomorphism from $\wedge^2\mathcal{F}$ to $\mathcal{F}$ and l_2 a homomorphism from $\wedge^2 X_0$ to X_0 such that $[[x_1],[x_2]] = [l_2(x_1,x_2)]$.

Lemma 1.2 *The bracket $[\cdot,\cdot]$ is well defined and a Lie bracket iff $\forall\ x_1, x_2, x_3 \in X_0$, $\forall\ y_1 \in X_1$, $\exists\ y_2, y_3 \in X_1$ such that*

$$l_2(l_1 y_1, x_1) = l_1 y_2 \tag{1.14}$$

$$l_2(l_2(x_1,x_2),x_3) + cyclic = l_1 y_3. \tag{1.15}$$

Proof. The proof is straightforward and can be found in [BFLS]. □.

Let sX_* be the graded vector space defined by $(sX)_k = X_{k-1}$. Let $T^c sX_* = \oplus_{k=0} sX_*^{\otimes^k}$ be the tensor coalgebra of sX_* with standard diagonal

$$\Delta(sx_1 \otimes \dots \otimes sx_k) = \Sigma_{j=0}^k (sx_1 \otimes \dots \otimes sx_j) \otimes (sx_{j+1} \dots \otimes sx_k). \tag{1.16}$$

Let $\bigwedge sX_*$ be the graded symmetric subcoalgebra of $T^c X_*$. Any homomorphism f from $\bigwedge^k sX_*$ to sX_* can be extended in a unique way as a coderivation to $\bigwedge sX_*$ (see for instance [LaSt, LaMa, Kje] for more details). Let us suppose this has been done in particular for l_1 and let us denote the resulting coderivation by $\hat{l}_1$. Let us define the total degree tot or (just degree for short) in $T^c sX_*$ of the element $sx_1 \wedge \dots \wedge sx_r$ to be $\Sigma_{i=1}^r |sx_i| = \Sigma_{i=1}^r |x_i| + r$, while the resolution degree res of this element is defined to be $tot - r = \Sigma_{i=1}^r |x_i|$.

These definitions imply that $\hat{l}_1$ is of resolution degree and total degree -1. Let $[\cdot,\cdot]$ be the graded commutator of coderivations with respect to the total degree.

Lemma 1.3 *The map l_2 can be extended as a degree -1 coderivation $\hat{l}_2$ of resolution degree* 0 *on $\bigwedge sX_*$ in such a way that $[\hat{l}_1, \hat{l}_2] = 0$.*

Proof. It is enough to verify the statement for $sx_1 \wedge sx_2$. For $x_1, x_2 \in X_0$, i.e, $sx_1 \wedge sx_2$ of resolution degree 0, the lemma is true since $\hat{l}_1$ vanishes on elements of resolution degree 0 and $res(l_2(sx_1 \wedge sx_2)) = 0$. For $res(sx_1 \wedge sx_2) = 1$, we can assume without loss of generality that $res(sx_1) = 1$, $res(sx_2) = 0$. We have $l_2\hat{l}_1(sx_1 \wedge sx_2) = l_2(l_1 sx_1, sx_2) + (-)^{|sx_1|} l_2(sx_1, l_1 sx_2) = l_2(l_1 sx_1, sx_2) = -l_1 sy$ because of (1.14). Hence, we can achieve $[\hat{l}_1, \hat{l}_2] = 0$ in resolution degree 1 if we define $l_2(sx_1 \wedge sx_2) = sy$. Suppose the lemma is true for $res(sx_1 \wedge sx_2) = k \geq 1$. For $res(sx_1 \wedge sx_2) = k+1$, we have $0 = l_2\hat{l}_1(\hat{l}_1(sx_1 \wedge sx_2)) = -l_1 l_2(\hat{l}_1(sx_1 \wedge sx_2))$. Since $res(l_2(\hat{l}_1(sx_1 \wedge sx_2))) > 0$, acyclicity of l_1 implies that there exists sy such $l_2(\hat{l}_1(sx_1 \wedge sx_2)) = -l_1(sy)$. Hence we can define $l_2(sx_1 \wedge sx_2) = sy$ in resolution degree $k+1$. $\square$.

Lemma 1.4 *There exists a degree -1 coderivation $\hat{l}_3$ of resolution degree* 1 *on $\bigwedge sX_*$ such that $\hat{l}_1\hat{l}_3 + \hat{l}_2\hat{l}_2 + \hat{l}_3\hat{l}_1 = 0$.*

Proof. It is enough to verify the relation for $sx_1 \wedge sx_2 \wedge sx_3$. In resolution degree 0, $\hat{l}_2\hat{l}_2(sx_1 \wedge sx_2 \wedge sx_3)$ is another expression for the left hand side of the Jacobi identity for the bracket l_2. Using (1.15), there exists l_3 such that $(l_1 l_3 + \hat{l}_2\hat{l}_2)(sx_1 \wedge sx_2 \wedge sx_3) = 0$. Since we are in resolution degree 0, this can also be written as $(l_1 l_3 + \hat{l}_2\hat{l}_2 + l_3\hat{l}_1)(sx_1 \wedge sx_2 \wedge sx_3)$. Let us suppose that the relation holds for $sx_1 \wedge sx_2 \wedge sx_3$ of resolution degree $k \geq 1$. If $sx_1 \wedge sx_2 \wedge sx_3$ is of resolution degree $k+1$, then $\hat{l}_1(sx_1 \wedge sx_2 \wedge sx_3)$ is of resolution degree k and hence $0 = (l_1 l_3 + \hat{l}_2\hat{l}_2 + l_3\hat{l}_1)\hat{l}_1(sx_1 \wedge sx_2 \wedge sx_3) = l_1(l_3\hat{l}_1 + \hat{l}_2\hat{l}_2)(sx_1 \wedge sx_2 \wedge sx_3)$. Acyclicity of l_1 in resolution degree > 0 then implies that one can extend l_3 to resolution degree $k+1$ in such a way that $\hat{l}_1\hat{l}_3 + \hat{l}_2\hat{l}_2 + \hat{l}_3\hat{l}_1 = 0$ holds. $\square$.

Lemma 1.5 *Acyclicity of l_1 in resolution degree > 0, i.e., $l_1(sx) = 0 \iff sx = l_1 sy$ for $res(sx) > 0$, implies acyclicity of $\hat{l}_1$ in the space of coderivations of resolution degree > 0 : for any $\hat{d}$ with $res(\hat{d}) = k > 0$ such that $[\hat{l}_1, \hat{d}] = 0$, there exists $\hat{t}$ with $res(\hat{t}) = k+1$ such that $\hat{d} = [\hat{l}_1, \hat{t}]$.*

Proof. Let ρ be a contracting homotopy for l_1, i.e., $l_1\rho + \rho l_1 = Id$ when applied to elements of resolution degree > 0. Without loss of generality, we can assume that $\hat{d}$ is the extension as a coderivation of a homomorphism $d : sX_*^{\bigwedge^k}$ to sX_*. Hence, we have on $sX_*^{\bigwedge^k}$, $d = (l_1\rho + \rho l_1)d$. The cocycle condition then implies that $d = l_1\rho d - \rho d\hat{l}_1 = [\hat{l}_1, \rho d]$. $\square$.

Theorem 1.1 *There exist degree* -1 *coderivations* $\hat{l}_k$ *of resolution degrees* $k-2$ *for* $k \geq 4$ *such that the degree* -1 *coderivation* $s = \hat{l}_1 + \hat{l}_2 + \hat{l}_3 + \hat{l}_4 + \dots$ *is a differential.*

Proof. The condition $\frac{1}{2}[s,s] = 0$ at resolution degree $k-1$ gives $I_k = \hat{l}_1\hat{l}_k + \hat{l}_2\hat{l}_{k-1} + \dots + \hat{l}_k\hat{l}_1 = 0$. We have shown above that this relation holds for $k \leq 3$. Suppose that it holds for all $l \leq k$, with $k \geq 3$. The graded Jacobi identity for $[\cdot,\cdot]$ implies that $\frac{1}{2}[s,[s,s]] = 0$. At resolution degree k, this identity reads $[\hat{l}_1, I_{k+1}] + [\hat{l}_2, I_k] + \dots + [\hat{l}_{k+1}, I_1] = 0$. The recursion hypothesis then implies that $[\hat{l}_1, \hat{l}_1\hat{l}_{k+1} + \hat{l}_2\hat{l}_k + \dots \hat{l}_k\hat{l}_2 + \hat{l}_{k+1}\hat{l}_1] = 0$. The first term cancels with the last one, so that $[\hat{l}_1, \hat{l}_2\hat{l}_k + \dots \hat{l}_k\hat{l}_2] = 0$. Since $k \geq 3$, the resolution degree of $\hat{l}_2\hat{l}_k + \dots \hat{l}_k\hat{l}_2$ is ≥ 1 so that the previous lemma implies the existence of $\hat{l}_{k+1}$ such that $\hat{l}_2\hat{l}_k + \dots \hat{l}_k\hat{l}_2 = -[\hat{l}_1, \hat{l}_{k+1}]$. $\square$. The associated skew coderivations $\tilde{l}_k$ on the skew coalgebra $\bigwedge_s X_*$, (see [LaSt, LaMa, Kje] for details) form an sh Lie algebra.

1.5 Reduced form

In the construction above, only l_1 and l_2 were initially fixed on X_0. In the extension of l_2 to $sX_*^{\bigwedge^2}$, and the construction of $l_3, l_4, \ldots$, we have the liberty, at each stage where we made a choice, to add l_1 exact terms. We will consider sh Lie algebras constructed in this way and corresponding to different choices to be equivalent.

In the case of local functionals, the resolution is trivial for resolution degree $> n$, which means horizontal form degree < 0. This means that the construction yields non vanishing brackets at most up to the bracket l_{n+2}.

Furthermore, the bracket l_2 on X_0 is such that (1.14) holds with 0 on the right hand side. This is because the bracket is expressed in terms of Euler-Lagrange derivatives which annihilate d_H exact n-forms. The following is due to M. Markl [Mar]:

Theorem 1.2 *If (1.14) holds with* 0 *on the right hand side, the extension of* l_2, l_3 *to* X_* *can be taken to be trivial. The sh Lie algebra is equivalent to an sh Lie algebra where only the brackets* l_1, l_2, l_3 *are non vanishing, i.e.,* $s = \hat{l}_1 + \hat{l}_2 + \hat{l}_3$ *is a differential.*

Proof. In the proof of lemma 1.3, we can take $sy = 0$ so that we can define $l_2(sx_1 \wedge sx_2) = 0$ if the resolution degree of $sx_1 \wedge sx_2$ is 1 and satisfy $[\hat{l}_1, \hat{l}_2] = 0$, i.e., $\hat{l}_1\hat{l}_2 = 0$ and $\hat{l}_2\hat{l}_1 = 0$. If we extend l_2 to be zero in all resolution degrees > 1, these two relations continue to hold on all of $sX_*^{\bigwedge^2}$. We have through (1.15) that there exist l_3 on $sX_*^{\bigwedge^3}$ such that $l_1 l_3 + \hat{l}_2\hat{l}_2 = 0$ on elements in resolution degree 0. We can choose l_3 to be zero on $\hat{l}_1$-exact terms in resolution degree 0 and in resolution degree > 0, so that $\hat{l}_1\hat{l}_3 + \hat{l}_2\hat{l}_2 + \hat{l}_3\hat{l}_1 = 0$ is non trivial only in resolution degree 0 where it reduces to $l_1 l_3 + \hat{l}_2\hat{l}_2 = 0$. In resolution degree 3, we have to consider the expression $\hat{l}_3\hat{l}_2 + \hat{l}_2\hat{l}_3$ which reduces to $\hat{l}_3\hat{l}_2$ because $\hat{l}_2$ vanishes on elements of resolution degree > 0. Let ρ be a contracting homotopy for l_1. We have $l_3 = (l_1\rho + \rho l_1)l_3 = l_1\rho l_3 - \rho l_2\hat{l}_2$. Hence, we can choose $l_3 = -\rho l_2\hat{l}_2$. The identity $[\hat{l}_1, \hat{l}_3\hat{l}_2 + \hat{l}_2\hat{l}_3] = 0$ reduces to $0 = \hat{l}_1\hat{l}_3\hat{l}_2 = \hat{l}_2\hat{l}_2\hat{l}_2$ because of our assumptions on $\hat{l}_2, \hat{l}_3$, so that $l_3\hat{l}_2 = -\rho l_2\hat{l}_2\hat{l}_2 = 0$. We then can take $\hat{l}_4$ and all higher order coderivations to be zero and $s = \hat{l}_1 + \hat{l}_2 + \hat{l}_3$ is a differential. □.

1.6 Sh Poisson algebra ?

In the above considerations, we have only been concerned with the Lie algebra aspects of Poisson brackets in field theory. The Poisson brackets in mechanics are in addition derivations with respect to the product of functions. The problem for Poisson brackets in field theory is that there is no product for local functionals. There is however a well defined product on Ω^n, but it does not induce a product on $\mathcal{F}$. In coordinates, it is defined by $L_1\nu \cdot L_2\nu = L_1 L_2\nu$. This product can be expressed in an invariant way on a Riemannian manifold in terms of the Hodge star and the wedge product. The brackets (1.11) satisfy the Leibnitz rule up to homotopy. Indeed, the skew adjointness of the operator ω^{ab} implies that

$$\{L_1\nu, L_2\nu \cdot L_3\nu\} = \omega^{ab}(\frac{\delta L_1}{\delta u^a})\frac{\delta L_2 L_3}{\delta u^b}\nu + d_H R_1$$

$$
\begin{aligned}
&= D_I[\omega^{ab}(\frac{\delta L_1}{\delta u^a})]\frac{\partial(L_2L_3)}{\partial u^b_I}\nu + d_H R_2 \\
&= \{L_1\nu, L_2\nu\}\cdot L_3\nu + \{L_1\nu, L_3\nu\}\cdot L_2\nu + d_H R_3,
\end{aligned}
\tag{1.17}
$$

for some $R_i \in \Omega^{n-1}$ obtained by integrations by parts. This suggests that in terms of representatives, i.e., elements of Ω^n, there should not only be a sh Lie structure, but rather an appropriately defined sh Poisson structure.

A different approach would be the following. The properties of a Poisson bracket in mechanics, including the Leibnitz rule with respect to the product, can be summarized by defining the bracket in terms of a two vector whose Schouten bracket with itself vanishes. In field theory, the situation is very similar, the Poisson bracket as defined in (1.11) is in fact associated with a functional two vector. Indeed, as mentioned before, one can show [Olv] that a functional two vector is uniquely determined by a skew adjoint operator ω^{ab}. Because the Schouten bracket for multi-vector fields together with the ordinary product for functions define a Gerstenhaber algebra, we can expect an sh Gerstenhaber algebra, which have been used recently in different contexts [LiZu, GeVo, KVZ], to be useful here as well.

2 Batalin-Vilkovisky bracket and Poisson bracket

2.1 The local antibracket in cohomology

In the Batalin-Vilkovisky formalism, we consider instead of the exterior algebra over dx^i with coefficients belonging to $Loc(E)$, the space $Loc(E)\otimes \Lambda(C^\alpha_I, u^*_{aI}, C^*_{\alpha I}, dx^i)$, where $C^\alpha_I, u^*_{aI}, C^*_{\alpha I}$ are the jet-bundle analogs of the ghosts, the antifields, the antighosts and their derivatives. The original fields u^a_I and the coordinates x^i with their differentials dx^i are defined to be of ghost number 0, while $C^\alpha_I, u^*_{aI}, C^*_{\alpha I}$ are respectively of ghost number 1, -1 and -2. If we define the fields $\phi^A \equiv (u^a_I, C^\alpha_I)$, the total derivative is extended to the new generators and reads:

$$
D_i = \frac{\partial}{\partial x^i} + \phi^A_{Ii}\frac{\partial}{\partial \phi^A_I} + \phi^*_{AIi}\frac{\partial}{\partial \phi^*_{AI}}. \tag{2.18}
$$

The BRST differential is denoted by s ; it is of ghost number 1 and satisfies $[s, d_H] = 0$, so that we have the bicomplex $(\Omega^{*,*}, s, d_H)$. The graded com-

mutator of graded derivations involves the total degree which is the ghost number plus the horizontal form degree.

The bracket is the BV antibracket, adapted from its functional expression to the n-forms as in the first part : it is given by

$$\{L_1\nu, L_2\nu\} = [\frac{\delta^R L_1}{\delta\phi^A}\frac{\delta^L L_2}{\delta\phi^*_A} - \frac{\delta^R L_1}{\delta\phi^*_A}\frac{\delta^L L_2}{\delta\phi^A}]\nu. \tag{2.19}$$

It satisfies a graded version of antisymmetry and Jacobi identity up to exact terms because it is expressed in Darboux coordinates so that the bracket induced in $H^{*,n}(d_H)$ is a graded Lie bracket corresponding to the usual Batalin-Vilkovisky bracket for local functionals.

The important property of the BRST differental s is that it is canonically generated through a variant of the local antibracket by a local functional $S = [L\nu]$ which is a solution of the BV master equation

$$\{S, S\} = 0 \Longleftrightarrow \{L\nu, L\nu\} = d_H R \tag{2.20}$$

for $R \in \Omega^{*,n-1}$,

$$s = \{L\nu, \cdot\}_{alt} = D_I(\frac{\delta^R L}{\delta\phi^A})\frac{\partial^L}{\partial\phi^*_{AI}} - D_I(\frac{\delta^R L}{\delta\phi^*_A})\frac{\partial^L}{\partial\phi^A_I}. \tag{2.21}$$

A proper solution of the master equation, in the case of an irreducible gauge theory, is of the form

$$S = \int_M (L_0 + u^*_a R^{aI}_\alpha D_I C^\alpha + ...)\nu, \tag{2.22}$$

where L_0 is the starting point Lagrangian, $R^{aI}_\alpha D_I$ describe a generating set of gauge symmetries and the higher terms are determined by requiring that S is a solution to the master equation.

The local antibracket (2.19) also induces a well defined bracket in $H^*(s, \mathcal{F}) \simeq H^*(s, H^n(d_H)) \simeq H^{*,n}(s|d_H)$. More precisely,

$$\begin{aligned} \{\cdot,\cdot\} : H^{g_1,n}(s|d_H) \times H^{g_2,n}(s|d_H) &\longrightarrow H^{g_1+g_2+1,n}(s|d_H) \\ \{[L_1\nu],[L_2\nu]\} &= [\{L_1\nu, L_2\nu\}]. \end{aligned} \tag{2.23}$$

The subspace $H^{-1,n}(s|d_H)$ equipped with the antibracket defines a subalgebra of $H^{*,n}(s|d)$ which we denote by $\mathcal{S}$,

$$\mathcal{S} = (H^{-1,n}(s|d), \{\cdot,\cdot\}). \tag{2.24}$$

The BRST cohomology groups contain the physical information of the problem. It is this last bracket, induced by the local antibracket in the local BRST cohomology groups $H^{*,n}(s|d_H)$ that we will relate to the Poisson bracket. It has been shown in [BBH], that the cohomology group in negative ghost numbers g describe the characteristic cohomology of the problem, i.e., the $n+g$ forms which are closed (under d_H) on the stationary surface Σ, which is defined by $D_I \frac{\delta L_0}{\delta u^a} = 0$ in the jet-bundle, without being exact on this surface. In positive ghost number, the cohomology of s describes equivariant cohomology associated to the gauge transformations on the stationary surface. The induced antibracket provides these cohomology groups with a graded Lie algebra structure. One can relate this Lie algebra structure in general with the graded Lie algebra induced by the Batalin-Fradkin-Vilkovisky extended Poisson bracket induced in the local Hamiltonian BRST cohomology groups [BaHe]. Since this is rather technical we will consider below only the more transparent cases of ordinary mechanics and non degenerate field theories.

2.2 Ordinary mechanics

The simplest case is ordinary mechanics in Hamiltonian form. The first order action

$$S = \int dt\ \dot{q}p - H \tag{2.25}$$

is by itself a proper solution to the master equation since there are no gauge invariances. The BRST differential is given by

$$s = (D_t)^k(\dot{q} - \frac{\partial H}{\partial p})\frac{\partial}{\partial p^{*(k)}} + (D_t)^k(-\dot{p} - \frac{\partial H}{\partial q})\frac{\partial}{\partial q^{*(k)}}, \tag{2.26}$$

where $p^{*(k)}, q^{*(k)}$ denote the k-th derivatives of the antifields p^*, q^* with respect to time t. We have to compute $H^{*,1}(s|d_H)$. This group is computed by the descent equations technique, so that we must first compute $H^*(s)$. It is straightforward to see that the time derivatives of p, q of order at least equal to 1 and the antifields with all their derivatives can be eliminated form the cohomology which is equal to $C^\infty(p,q)$. Let $f \in C^\infty(p,q)$. We have to solve $sgdt + d_H f = 0$ in ghost number -1. This equation implies that $g = dt[\frac{\partial f}{\partial q}p^* - \frac{\partial f}{\partial p}q^*]$ with f satisfying $[f,H]_P = 0$, where $[\cdot,\cdot]_P$ is the standard

Poisson bracket. In other words, the bottom f of the descent equations is a first integral, while the corresponding top in ghost number -1 is the associated Hamiltonian vector field upon identification of q^*, p^* with $\frac{\partial}{\partial q}, \frac{\partial}{\partial p}$. This identification implies that the Lie algebra $\mathcal{S}$ is isomorphic to the Lie algebra of Hamiltonian vector fields for first integrals equipped with the standard Lie bracket for vector fields:

$$\{\frac{\partial f_1}{\partial q}p^* - \frac{\partial f_1}{\partial p}q^*, \frac{\partial f_2}{\partial q}p^* - \frac{\partial f_2}{\partial p}q^*\} = \frac{\partial [f_1, f_2]_P}{\partial q}p^* - \frac{\partial [f_1, f_2]_P}{\partial p}q^*. \quad (2.27)$$

This last algebra is isomorphic to the Poisson algebra of first integrals, which gives the desired relation between the local antibracket induced in cohomology and the Poisson bracket.

2.3 Non degenerate field theories

In a field theory without gauge symmetries, $S = \int L_0 \nu$ is a proper solution to the master equation and the BRST symmetry reduces to

$$s = D_I(\frac{\delta^R L_0}{\delta u^a})\frac{\partial^L}{\partial u^*_{aI}} \equiv \delta, \quad (2.28)$$

which is called the Koszul(-Tate) differential associated to the stationary surface Σ. Its cohomology in the algebra $Loc(E) \otimes \Lambda(u^*_{aI}, dx^i)$ is given by $H^0(\delta) \simeq Loc(E)/I \otimes \Lambda(dx^i)$, where I is the ideal of horizontal forms vanishing on Σ, while $H^k(\delta) = 0$ for $k < 0$. We are interested in $H^{-1,n}(\delta|d_H)$ in the algebra $Loc(E)/bfR \otimes \Lambda(u^*_{aI}, dx^i)$.

Theorem 2.3 (Cohomological formulation of Noether's first theorem)

$$H^{-1,n}(\delta|d_H) \simeq H^{n-1,0}(d_H|\delta). \quad (2.29)$$

Proof. A cocycle in the first group satisfies $\delta a + d_H j = 0$. If we consider a different representative $a' = a + \delta() + d_H()$, the corresponding j is modified only by δ and d_H exact terms. Hence, the map $f([a]) = [j]$ from $H^{-1,n}(\delta|d_H)$ to $H^{0,n-1}(d_H|\delta)$ is well defined. It is injectif because if $[j] = 0$, i.e., $j = d_H() + \delta()$, then acyclicity of δ in ghost number -1 implies that $a = \delta() + d_H()$. Inverting the role of δ and d_H and using the acyclicity of d_H in form

degree $n-1$, we find that the map f is bijective. $\square$. The interpretation of $H^{n-1,0}(d_H|\delta)$ is clear. Since δ exact forms in ghost number 0 are precisely given by forms which vanish on Σ, this group describes the characteristic cohomology in form degree $n-1$, or in dual language, the equivalence classes of currents that are conserved on the stationary surface, $D_i j^i = 0$ on Σ, where two currents are equivalent if they differ by a current of the form $D_k S^{[ki]}$ on Σ, with $j^i, S^{[ki]} \in Loc(E)$.

By allowed redefinitions, (adding d_H exact terms, or in dual notation, doing integrations by parts), we can assume that a representative of an element of $H^{-1,n}(\delta|d_H)$ is of the form $a = u^*_a X^a \nu$, with $X^a \in Loc(E)$. The cocycle condition reads

$$\frac{\delta L_0}{\delta u^a} X^a + D_i j^i = 0, \tag{2.30}$$

for some $j^i \in Loc(E)$. Hence, the evolutionary vector field $X^a \frac{\partial}{\partial u^a}$ defines a variational symmetry of L_0. The coboundary conditions implies that a trivial variational symmetry is of the form $X^a u^*_a = \delta(\frac{1}{2} u^*_{bJ} u^*_{aI} \mu^{[aIbJ]}) + D_i k^i$, for some $\mu^{[aIbJ]} \in Loc(E)$ which are antisymmetric under the exchange $aI \longleftrightarrow bJ$ and some k^i which are linear polynomials in u^*_{aI}. Taking Euler-Lagrange derivatives with respect to u^*_a, this implies $X^a = (-D)_I [D_J \frac{\delta L_0}{\delta u^b} \mu^{[aIbJ]}]$. In particular, X^a must vanish on the stationary surface.

Hence, the lemma above expressed that there is an isomorphism between equivalence classes of variational symmetries (two variational symmetries are equivalent if they differ by a trivial symmetry defined above) and equivalence classes of conserved currents. This is precisely the content of Noether's first theorem.

If we identify u^*_a with $\frac{\partial}{\partial u^a}$, the antibracket $\{[a_1],[a_2]\}$ can be identified with the ordinary Lie bracket for variational symmetries induced by the Lie bracket for evolutionary vector fields,

$$[\{a_1, a_2\}] = [(\frac{\delta(u^*_a X^a_1)}{\delta u^b} X^b_2 - \frac{\delta(u^*_a X^a_2)}{\delta u^b} X^b_1)\nu = [u^*_a [X_2, X_1]^a \nu], \tag{2.31}$$

so that $\mathcal{S}$ is isomorphic to the Lie algebra of variational symmetries equipped with the bracket for evolutionary vector fields.

From the isomorphism, we know that there is an induced bracket in $H^{n-1,0}(d_H|\delta)$. An explicit calculation involving the properties of Euler-

Lagrange derivatives gives [BaHe]

$$\{[j_1],[j_2]\} = -[D_I(X_1^a)\frac{\partial}{\partial u_I^a}j_2] = [D_I(X_2^a)\frac{\partial}{\partial u_I^a}j_1]. \tag{2.32}$$

This bracket has been proposed by Dickey [Dic1] as a covariant way to define Poisson brackets for conserved currents, so that $\mathcal{S}$ is isomorphic to the Lie algebra of (equivalence classes of) conserved currents equipped with the Dickey bracket. Finally, one can show [Dic1, BaHe] that this Lie algebra is equivalent to the Lie algebra of conserved charges $Q = \int dx^1 \wedge \ldots \wedge dx^{n-1} j^0$ equipped with the local Poisson bracket induced in the space of functionals on space alone. This gives the desired relation between the antibracket and the Poisson bracket.

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